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A gravitational analogue of the Aharonov–Bohm effect

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Abstract. It is shown that particles constrained to move in a region where the Riemann tensor vanishes may nonetheless exhibit physical effects arising from non-zero curvature in a region from which they are excluded. This is a gravitational analogue of the Aharonov–Bohm effect.

The Aharonov-Bohm effect in electrodynamics can be interpreted as an illustration of non-locality in quantum theory. This effect, which was predicted theoretically by Aharonov and Bohm (1959) and confirmed experimentally by Chambers (1960), arises when a coherent beam of electrons is directed around either side of a solenoid within which a non-zero magnetic field is present. If the electron beams are allowed to interfere, it is found that the phase difference depends upon the magnetic flux passing through the surface bounded by the beam paths. This is still true even if the electrons are excluded from the region where the magnetic field is non-zero; hence the interpretation of non-locality. This effect can equally well be interpreted as illustrating the need to couple the electron to the electromagnetic vector potential. The phaseshift is proportional to the line integral of the vector potential around the path of the electron beam which by Stokes' theorem is equal to the enclosed magnetic flux. The vector potential in the region traversed by the electrons is pure gauge in the sense that in a local neighbourhood it is possible to find a gauge in which the vector potential is identically zero. However, there does not exist any gauge choice which makes $\boldsymbol{A}$ vanish everywhere along the path.

The question naturally arises as to whether an analogous effect exists in the theory of gravitation. The metric $g_{\mu\nu}$ and Riemann curvature tensor $R^{\mu}_{\ \alpha\nu\beta}$ play roles analogous to those of the potentials and field strengths respectively in electromagnetism. A genuine gravitational field is associated with a non-vanishing Riemann tensor, whereas a metric whose Riemann tensor vanishes may be transformed to the Minkowski metric by a coordinate transformation. However, the Aharonov-Bohm effect suggests that particles constrained to move in a region where the Riemann tensor vanishes may nonetheless exhibit physical effects arising from non-zero curvature in a region from which they are excluded. It is the purpose of this paper to show that this is indeed the case.

The Sagnac effect in general relativity has been investigated by Ashtekar and Magnon (1975), who point out that the gravitational field of a rotating mass distribution produces effects analogous to the Aharanov–Bohm effect. Their work has been extended by Anandan (1977). In the present paper, we shall be concerned with a somewhat different gravitational analogue to the Aharonov-Bohm effect which can arise even for a static mass distribution. A particular case was recently considered by Vilenkin (1981) in the context of a string model.

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Let us consider a space–time associated with a tube-like distribution of matter. We assume the existence of two Killing vector fields, t^μ which is time-like and z^μ which is space-like and generates translations along the direction of the tube. Thus the matter distribution is stationary and uniform along z^μ. We also assume that the space is asymptotically flat in a direction perpendicular to z^μ (i.e. at large distances from the tube). Let S denote a 2-surface which is orthogonal to both t^μ and z^μ. Because of the time and spatial translational symmetry, all such 2-surfaces are equivalent. We are assuming that the Riemann tensor approaches zero as one moves in S in directions away from the tube. Let C be a closed curve in S which lies entirely in the asymptotically flat region but which encloses the tube where the curvature is non-zero. If one transports a vector parallel to C around the closed curve, it will, in general, not return to itself but will undergo a rotation by an angle α which is expressible as the area integral of the Gaussian curvature K over S_0, the subsurface of S enclosed by C (Stoker 1969):

$$\alpha = \int_{S_0} K \, \mathrm{d}a. \tag{1}$$

This is one form of the Gauss–Bonnet theorem. Thus even though the curvature may vanish along C, the effects of non-zero curvature in the interior region are still felt. Geometrically, this means that S is asymptotically a conical surface rather than a plane. Physically, it means that particles may be gravitationally deflected by the tube without ever entering a region of non-zero curvature.

To understand this effect more explicitly, let us consider a static space-time described by the metric

$$\mathrm{d}s^2 = -D^2(r, \theta)\,\mathrm{d}t^2 + A^2(r, \theta)\,\mathrm{d}r^2 + B^2(r, \theta)\,\mathrm{d}\theta^2 + E^2(r, \theta)\,\mathrm{d}z^2 \tag{2}$$

which is translationally invariant in the t and z directions. (This form of the metric contains more functions than are essential; for example, by a suitable choice of r and θ one could require that $A = B$.) The 2-surface S is a surface defined by t and z equal constants and has the metric

$$\mathrm{d}s_S^2 = A^2(r, \theta)\,\mathrm{d}r^2 + B^2(r, \theta)\,\mathrm{d}\theta^2. \tag{3}$$

We now assume that the coordinates can be chosen so that $A \to 1$ and $B \sim r$ as $r \to 0$ and that $A \to 1$ and $B \to br$ as $r \to \infty$. Furthermore, for all values of r, the points θ and $\theta + 2\pi$ are identified. Thus, in a neighbourhood of the origin, S is flat (i.e. free of conical singularities) and for large r it becomes a cone.

The Gaussian curvature of S may be expressed in terms of A and B and their partial derivatives:

$$\begin{aligned} K &= A^{-3}B^{-3}(A^2 A_{,\theta} B_{,\theta} + B^2 A_{,r} B_{,r} - AB^2 B_{,rr} - A^2 B A_{,\theta\theta}) \\ &= -A^{-1}B^{-1}\left[\left(\frac{A_{,\theta}}{B}\right)_{,\theta} + \left(\frac{B_{,r}}{A}\right)_{,r}\right]. \end{aligned} \tag{4}$$

If we integrate K over S, using the above asymptotic forms of A and B, we find that

$$1 - b = \frac{1}{2\pi}\int K\sqrt{g}\,\mathrm{d}^2x \tag{5}$$

where $\sqrt{g} = AB$. This is equivalent to (1) with $\alpha = 2\pi(1-b)$. The surface S is, at large r, a cone with conical angle $\Delta = \pi b$. The conical angle of a cone is here defined as the angle subtended at the apex (so $\Delta = \pi$ for a plane).

The Gaussian curvature K of S may be readily expressed in terms of the Riemann tensor ${}^{(4)}R^{\mu}_{\alpha\nu\beta}$ of the four-dimensional space–time within which S is embedded. First consider a 3-surface Σ which is orthogonal to t^{μ}. The Riemann tensor ${}^{(3)}R^{i}_{jkl}$ of Σ is expressible in terms of ${}^{(4)}R^{\mu}_{\alpha\nu\beta}$ and the extrinsic curvature K_{ij} of Σ by means of the Gauss–Codazzi equations. However, because t^{μ} is a Killing vector, Σ has zero extrinsic curvature. This follows because (Misner *et al* 1973)

$$K_{ij} = -t_{i;j} \tag{6}$$

and is symmetric $K_{ij} = K_{ji}$. Killing's equation requires that $t_{i;j} = -t_{j;i}$ or $K_{ij} = -K_{ji}$ and hence $K_{ij} = 0$. In this case ${}^{(3)}R^{i}_{jkl}$ is obtained by projecting ${}^{(4)}R^{\mu}_{\alpha\nu\beta}$ into Σ, or in coordinates where Σ is defined by $t = \text{constant}$,

$${}^{(3)}R^{i}_{jkl} = {}^{(4)}R^{i}_{jkl}. \tag{7}$$

Similarly, one obtains that the Riemann tensor of the two-dimensional space S is

$${}^{(2)}R^{i}_{jkl} = {}^{(4)}R^{i}_{jkl}. \tag{8}$$

The Gaussian curvature is one half of the scalar curvature ${}^{(2)}R$ of S, and hence

$$K = \tfrac{1}{2}{}^{(2)}g^{ik}\,{}^{(2)}g^{jl}\,{}^{(4)}R_{ijkl} \tag{9}$$

where ${}^{(2)}g_{ij}$ is the metric on S. This result, combined with equation (5), enables us to find the conical angle in terms of the Riemann tensor of the four-dimensional space–time.

In general it does not seem to be possible to express K in terms of the four-dimensional Ricci tensor, and hence of the energy–momentum tensor of the source, $T_{\mu\nu}$. However, in the case that the gravitational field is sufficiently weak that the linearised theory may be applied, such an expression can be given. Let the metric be

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad \text{where } \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$$

and let

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}h \tag{10}$$

with $h = h^{\alpha}_{\alpha}$. With the gauge condition

$$\bar{h}^{\mu\nu}{}_{,\nu} = 0 \tag{11}$$

the linearised field equations become

$$\Box \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu} \tag{12}$$

and the Riemann tensor is

$${}^{(4)}R_{\alpha\mu\beta\nu} = \tfrac{1}{2}(h_{\alpha\nu,\mu\beta} + h_{\mu\beta,\alpha\nu} - h_{\mu\nu,\alpha\beta} - h_{\alpha\beta,\mu\nu}). \tag{13}$$

(Our notation is that of Misner *et al* (1973) with $G = c = 1$.) As before, the source is assumed to be independent of the x^0 and x^3 coordinates. Then

$$K = {}^{(4)}R_{1212} = \tfrac{1}{2}(h_{12,21} + h_{21,21} - h_{22,11} - h_{11,22}). \tag{14}$$

Using (11) and (12), this may be rewritten as

$$K = 8\pi(T_{11} + T_{22} - \tfrac{1}{2}T) \tag{15}$$

where $T = T^{\alpha}_{\alpha}$. This inserted into (5) yields b in terms of the source. Of particular interest is the case where $\rho = T_{00}$ is the only non-zero component of $T_{\mu\nu}$. Then we have

$$1-b = 4\pi \int \rho \, d^2x. \tag{16}$$

Thus for a dust source, the conical angle is given in terms of the mass per unit length of the tube.

Let us now consider the physical consequences of the fact that S is asymptotically a cone rather than a plane. One consequence follows immediately from (1). If a spinning particle travels around a closed curve in such a way that the spin vector undergoes parallel transport, the spin vector will be rotated by an angle α upon return to the starting point. This will effect the interference properties of a coherent beam of particles such as neutrons. (For a discussion of experiments using the interference properties of neutrons to measure gravitational and rotational effects, see Anandan (1977) and Greenberger and Overhauser (1979).) Neutrons which traverse a region of space where the curvature is identically zero are nonetheless capable of detecting the effects of curvature in other regions of space–time.

There is a second type of interference effect by means of which one could detect the region of non-zero curvature. Consider a circular loop of proper radius R which is centred about a region containing a source, with locally flat space outside of the source. If R is much larger than the dimensions of the source, then the circumference of the circle is $2\pi bR$. If one were to vary the characteristics of the source in such a way that b changes, but require that R remains fixed, the proper length of such a circular path would change. Hence, a coherent beam of light or other particles which travels around this path could be made to exhibit interference which reflects this varying path length.

In the case that the space–time is identically flat (rather than asymptotically flat) outside of the source, one can relate the conical angle to the deflection of a beam of particles (Vilenkin 1981). The metric in the flat region can be written as

$$ds^2 = -dt^2 + dr^2 + b^2 r^2 \, d\phi^2 + dz^2 \tag{17}$$

where $0 \leqslant \phi < 2\pi$, which is equivalent to

$$ds^2 = -dt^2 + dr^2 + r^2 \, d\phi'^2 + dz^2 \tag{18}$$

with $0 \leqslant \phi' < 2\pi b$. The path of a photon or other free particle moving tangentially is a straight line in the latter coordinates and sweeps out an angle $\Delta\phi' = \pi$ or

$$\Delta\phi = \pi b^{-1}. \tag{19}$$

Hence the particle undergoes a deflection of

$$\delta\phi = \pi(b^{-1} - 1) \tag{20}$$

due to the gravitational effects of the tube. Note that the deflection is independent of the impact parameter or of whether the particle travels along a time-like or null geodesic. In the case where the space–time is not identically flat along the particle's path, there is still a deflection but it is no longer given by (20).

The light deflection (20) can be transformed away: indeed, the metric (18) can be brought to the Galilean form

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

in any space–time region which does not contain closed loops enveloping the cylindrical source. However, such a transformation cannot be performed in the whole space–time.

Light beams propagating on different sides of the source can intersect, and thus the relative deflection of two beams cannot, in general, be transformed away. The situation here is analogous to the electromagnetic Aharonov–Bohm effect, where the vector potential can be transformed to zero in any region containing no closed loops enclosing the solenoid.

A difference between the electromagnetic Aharonov–Bohm effect and the gravitational analogue discussed here is that the former is a quantum interference effect, whereas the latter is classical (i.e. independent of $\hbar$). This is attributable to the fact that the classical equations of motion in the electromagnetic case involve only the field strength $F_{\mu\nu}$, so there can be no classical deflection in a region where the fields are zero. On the other hand, the equation of motion for a particle in a gravitational field, the geodesic equation, involves the metric rather than the Riemann tensor; consequently non-local effects of curvature can arise at the classical level.

Examples of space–times which are flat outside of the source include the vacuum strings (Vilenkin 1981). It is necessary for the source to exhibit either negative pressure or negative energy density, which do not occur in most familiar types of matter but do arise in both classical and quantum field theories.

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